Exceptional Dehn fillings and necklaces

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Basic definitions

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Definition

A *Dehn filling* of a link complement M is $M \cup_\phi \Phi$, where Φ is a disjoint union of solid tori, and $\phi : \partial\Phi \to \partial M$ is an orientation-reversing embedding.

The traditional Dehn filling cartoon

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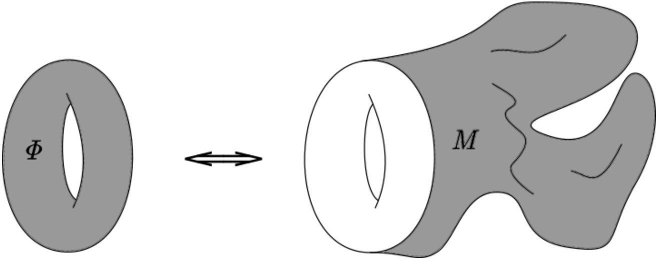

Hyperbolic manifolds

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Remark

Hyperbolic link complements have finite volume.

A conjecture

Theorem (Lackenby-Meyerhoff)

A hyperbolic knot complement has at most 10 nonhyperbolic "exceptional" Dehn fillings.

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A hyperbolic knot complement has at most 10 nonhyperbolic "exceptional" Dehn fillings.

Conjecture (C. McA. Gordon)

$S^3 \setminus 4_1$ is the unique hyperbolic knot complement admitting 10 exceptional Dehn fillings.

Cusp geometry

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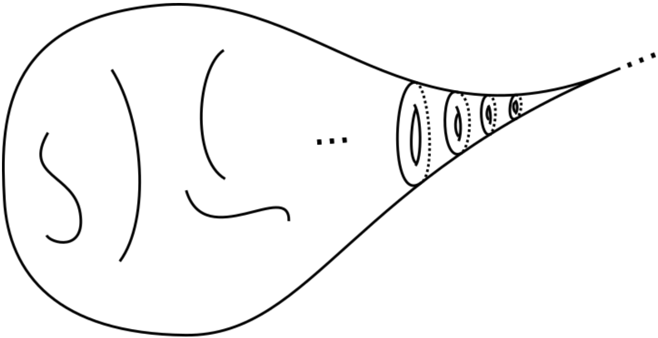

Maximal cusp objects

The union of these tubular neighborhoods is called the *maximal cusp neighborhood*. Its boundary is also a Euclidean torus, but with a finite set of points of self-tangency. This is called the *maximal cusp torus*, and its area is the *maximal cusp area* of M, or just the *area* of M.

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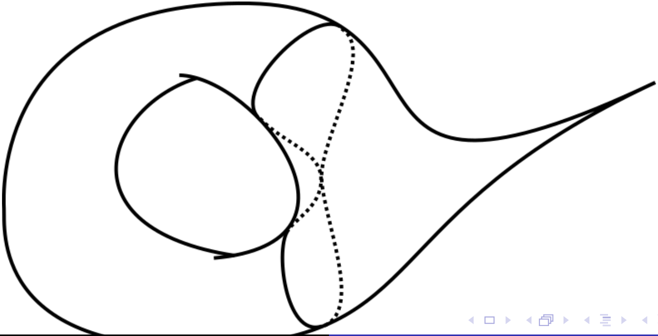

Big area implies few exceptional fillings

Remark (Agol)

If the maximal cusp area of a hyperbolic knot complement is at least 36/7, then it has fewer than 9 nonhyperbolic Dehn fillings.

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If the maximal cusp area of a hyperbolic knot complement is at least 36/7, then it has fewer than 9 nonhyperbolic Dehn fillings.

Unfortunately, there are infinitely many hyperbolic knot complements with maximal cusp area less than 36/7.

Partial picture near a maximal cusp

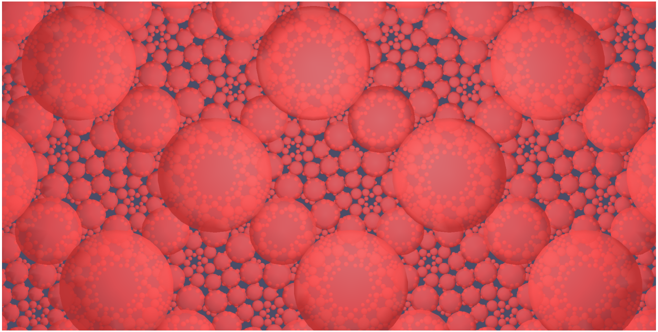

Figure: Some lifts of a maximal horoball neighborhood of a cusp.

Even more partial picture near a maximal cusp

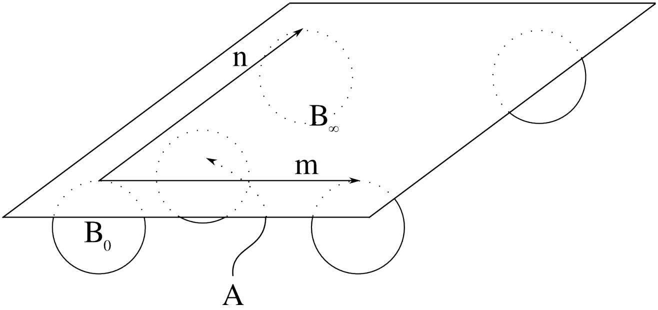

Figure: Fewer lifts of a maximal horoball neighborhood of a cusp, with labels.

Bicuspid groups

Definition

Let $BC = \langle m, n, g \mid [m, n] \rangle$. A *bicuspid group* is a representation $\phi : BC \to Isom^+(H^3) = PSL_2\mathbb{C}$ such that $\phi|_{\langle m,n \rangle}$ is injective and $trace(\phi(m)) = trace(\phi(n)) = \pm 2$.

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Definition

A *standard bicuspid group* is a bicuspid group ϕ such that

$$\phi(m) = \pm \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},\ \phi(n) = \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},\ \text{and } \phi(g) = \pm \begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix},$$

for numbers a, b, c such that $1 \leq |a|$, $|a| \leq |b|$, and $|c| \leq |b|$. The *area* of a standard bicuspid group is the area of the parallelogram spanned by a, b.

Agol on bicuspid groups

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- *Every discrete, torsion-free bicuspid group $\phi : BC \to PSL_2\mathbb{C}$ with infinite covolume is faithful.*
- *Every discrete, faithful standard bicuspid group has area at least $2 \cdot \pi$.*

Crash collars

Definition

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Definition

Let $BC^* = BC \setminus \langle m, n \rangle$. If $w \in BC^*$, then the *crash collar* associated to w is the subset of $\mathcal{P}$

$$C_w = \{\phi \in \mathcal{P} : \phi(w)(B_\infty) \cap B_\infty \neq \emptyset\},$$

where B_∞ is the "horoball at infinity"

$$B_\infty = \{(x, y, z) \in H^3 : z > 1\},$$

using the upper half space model of H^3.

Crash collars cover

Remark (Agol)

- *If* ϕ *is indiscrete, then there are words* $w \in BC^*$ *such that* $\phi(w)$ *is arbitrarily close to the identity. Thus there is some word* w *such that* $\phi(w)(B_\infty) \cap B_\infty$ *is nonempty.*

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- *On the other hand, if* ϕ *has discrete image, then by Agol's lemmas,* ϕ *either has finite covolume or has torsion. So then there is some word* w *in* m, n, g *such that* $\phi(w)(B_\infty) = B_\infty$.

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- *On the other hand, if ϕ has discrete image, then by Agol's lemmas, ϕ either has finite covolume or has torsion. So then there is some word w in m, n, g such that $\phi(w)(B_\infty) = B_\infty$.*
- *Therefore, the family of crash collars over all words in m, n, g forms an open cover of the compact set $\mathcal{P}$.*

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Construction (MOM Project)

Using a sophisticated procedure and several CPU months, we have found a finite list L *of words in* m, n, g *and a partition* $L = B \sqcup H$ *such that*

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N.B. $|H| = 86$. It's not too big.

Crash collars cover corollary

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Corollary (MOM Project)

For every hyperbolic knot complement N with 10 exceptional fillings, there is a bicuspid subgroup ρ of $\pi_1(N)$ and a word $h \in H$ such that $h \in \ker \rho$.

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For instance, $B_0 \succ\prec B_\infty$. Therefore, $gB_0 \succ\prec gB_\infty$. But $gB_0 = B_\infty$. Hence $B_\infty \succ\prec gB_\infty = A$, an *Adams horoball*.

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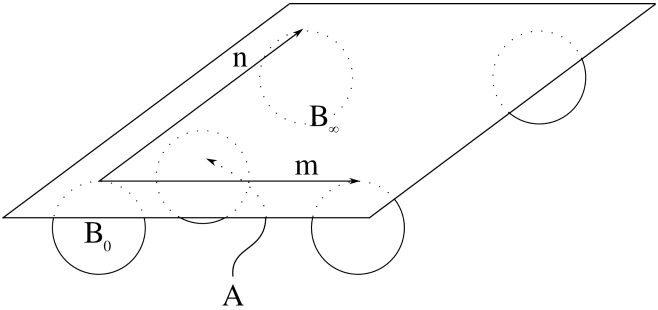

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What happens when you apply successive g-nemes to B_∞*?*

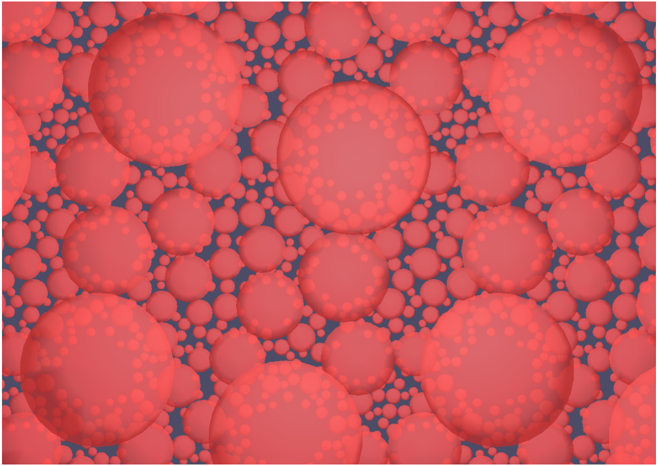
$$B_\infty \succ\prec gmGmnGmA \succ\prec gmGmnB_0 \succ\prec gmB_0 \succ\prec A \succ\prec B_\infty.$$

Necklace from quasirelator

Lemma (MOM Project)

Every hyperbolic knot complement N with 10 exceptional Dehn fillings has a "necklace" of **at most 7** *horoball "beads" in the lift of the maximal cusp neighborhood to the universal cover.*

The lifts for our example



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We want to show that the above lemma implies that every hyperbolic knot complement is a Dehn filling on a torus-based handle structure with one 1-handle and one 2-handle both of valence at most 7.

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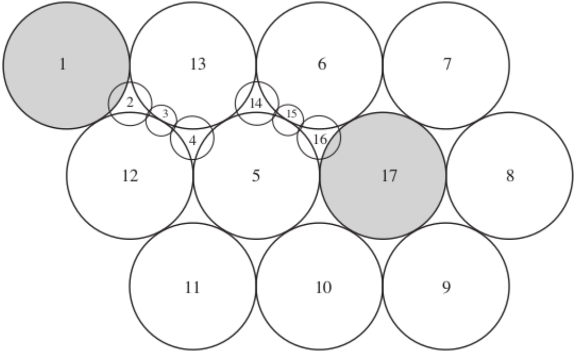

No small knots

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Lemma (MOM Project)

Necklaces with at most 8 beads are unknotted.

So we can find a disc in the universal cover along any necklace of 7 beads. But this disc might not descend to a disc in the knot complement.

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A disc on such a necklace will not descend to an embedded disc in the knot complement.

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Lemma (Adams-Knudsen)

b-bead necklaces are unblocked for $b \leq 6$.

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We're confident we can extend the lemma of Adams and Knudsen to $b = 7$, thus reducing Gordon's conjecture to a finite enumeration of simple handle structures and their exceptional Dehn fillings.

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Thank you for your time!