On the complexity of cusped non-hyperbolicity

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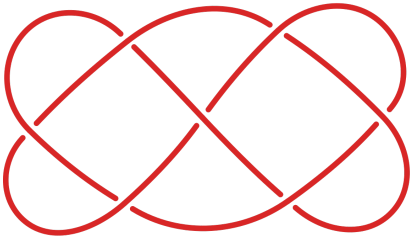
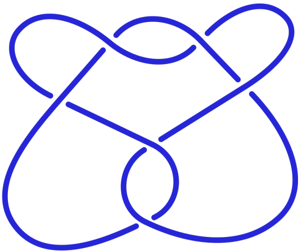
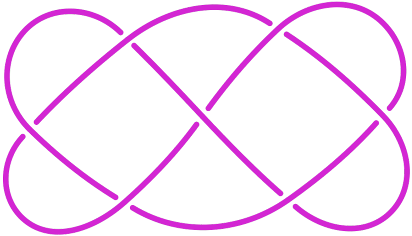
Applications and Computations in Knot Theory

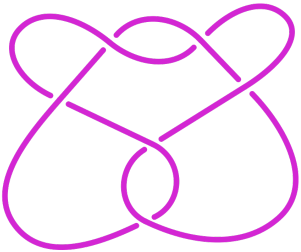
Theorem (Haraway-Hoffman '19)

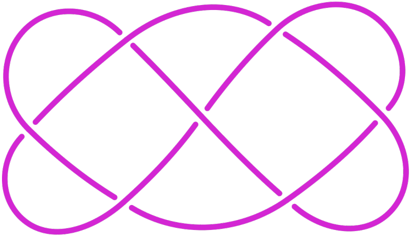
For links in S^3, hyperbolicity is in coNP.



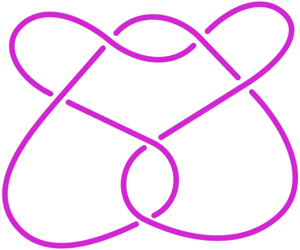


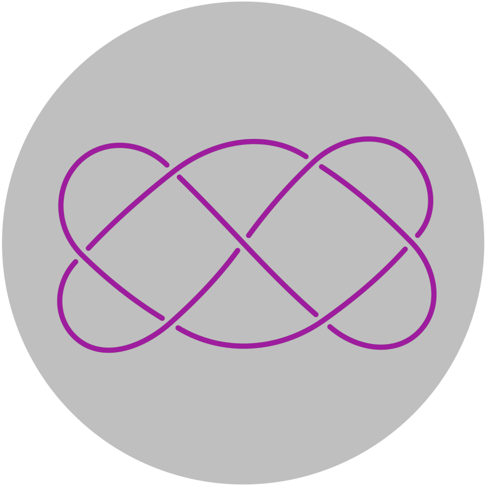


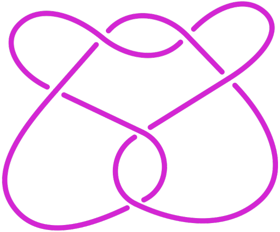


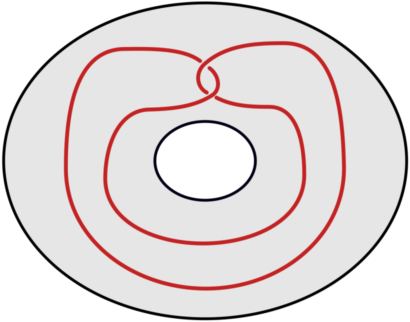


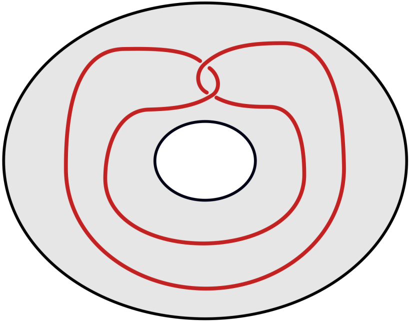
Split

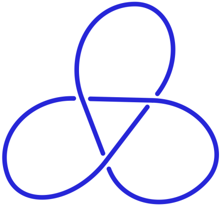


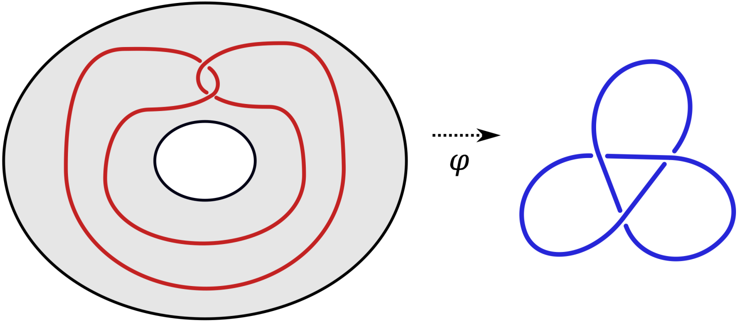


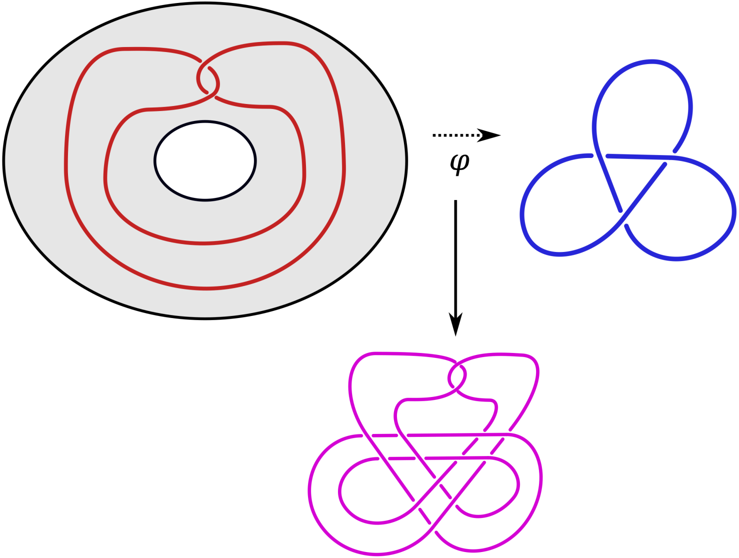


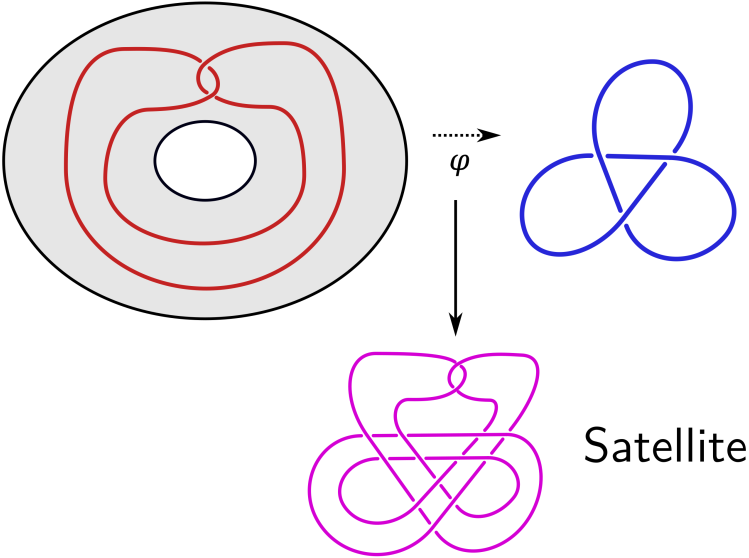


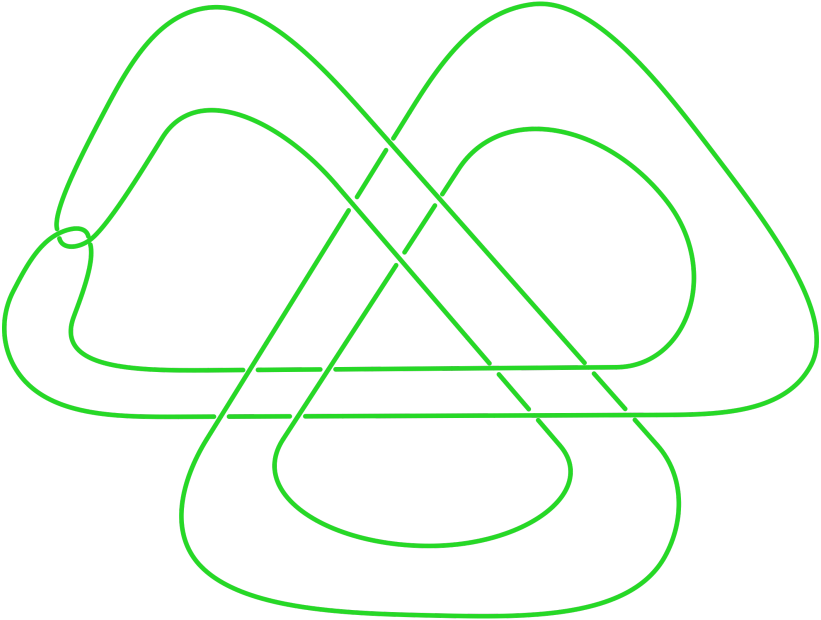


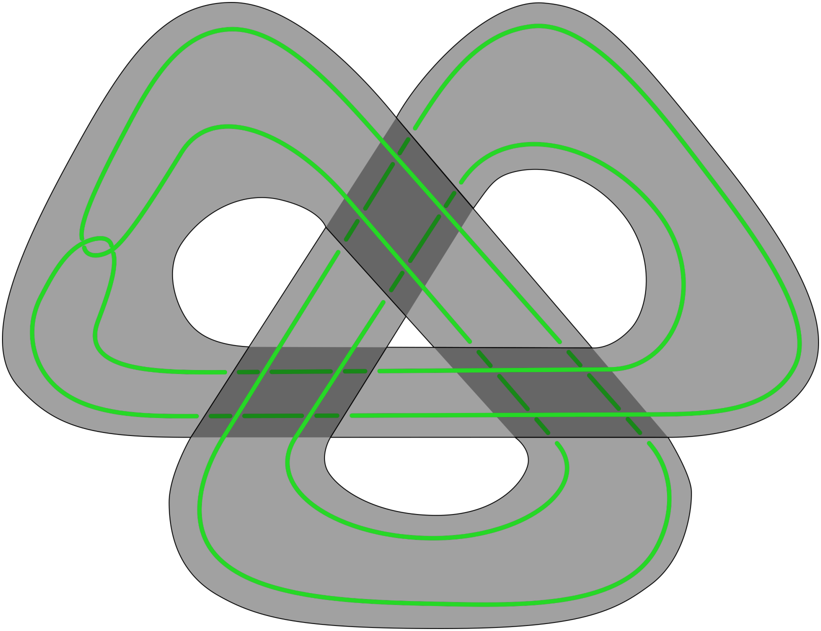


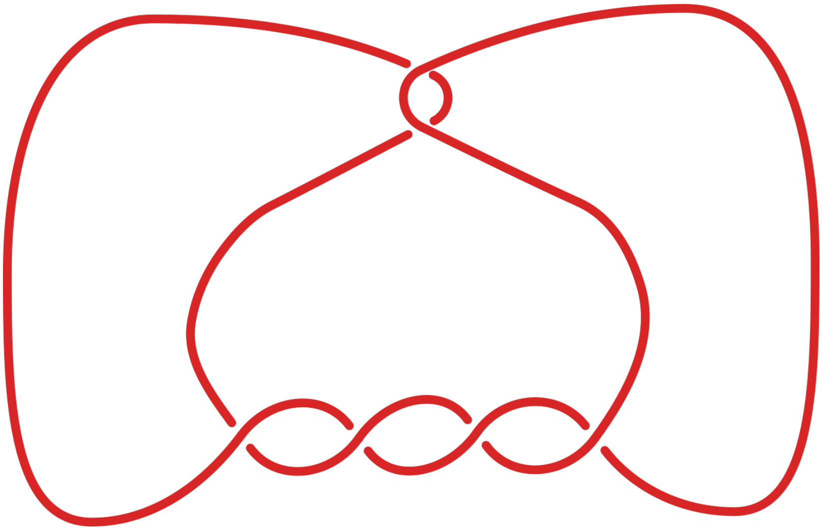

φ

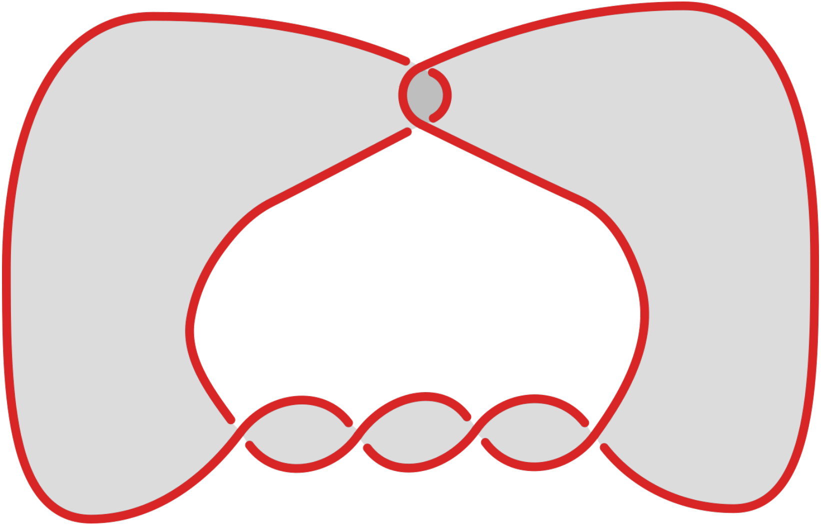

φ


φ
Satellite

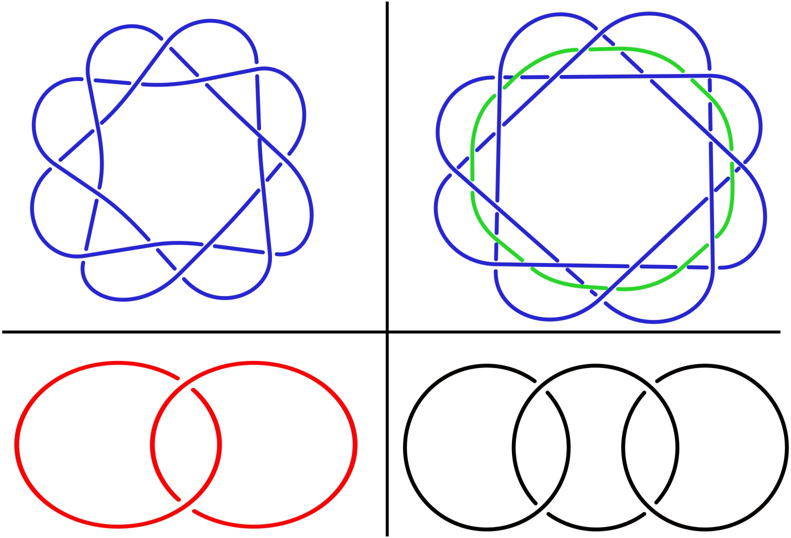








Annular links



Theorem (Thurston '82)

A link in S^3 is either the unknot, split, satellite, annular, or

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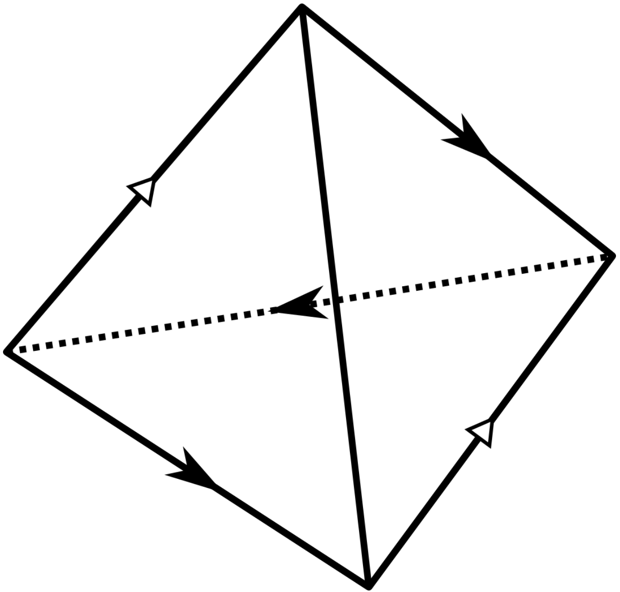
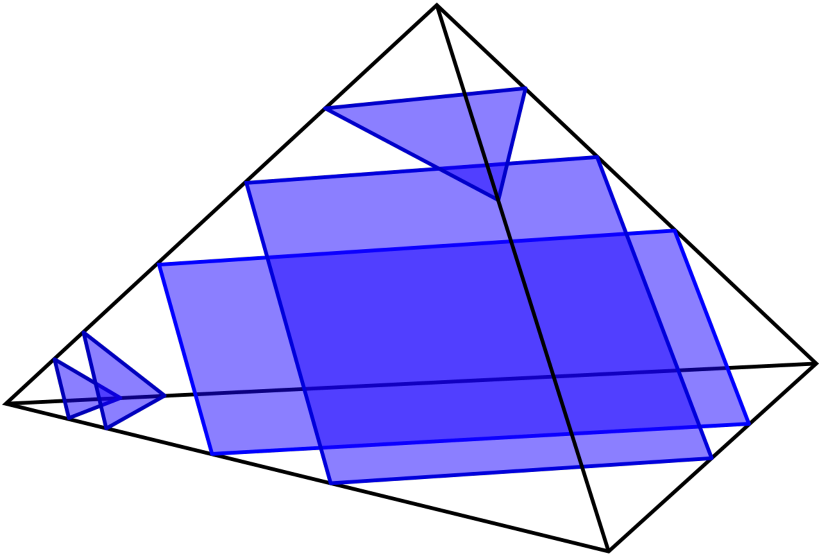
A link in S^3 is either the unknot, split, satellite, annular, or ***hyperbolic!***

Solid torus



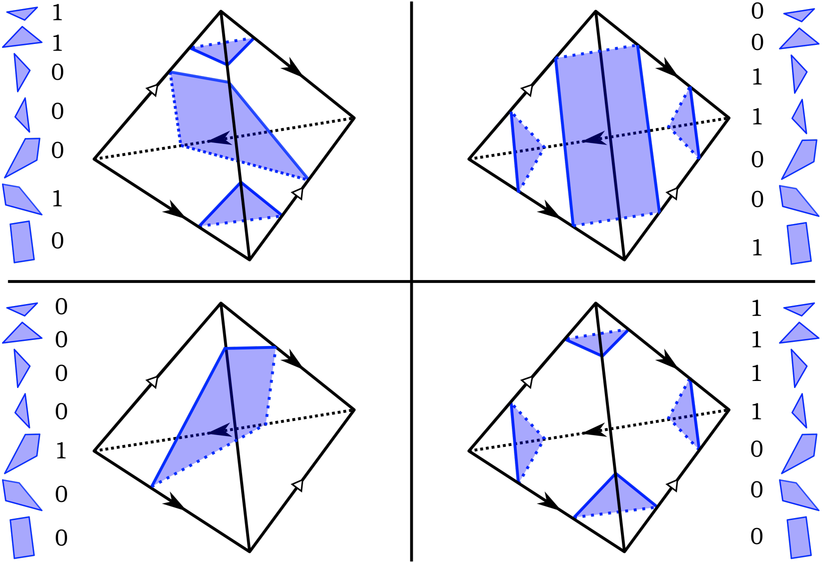
Normal discs



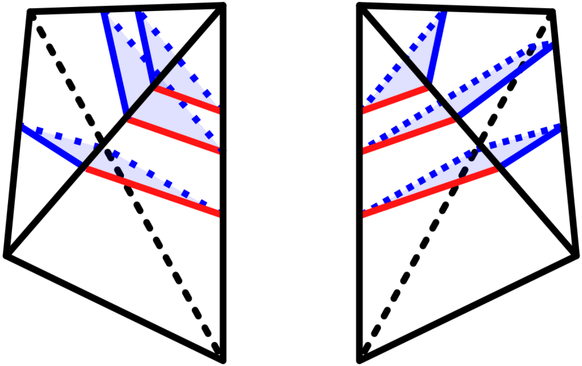
Fact

If there is an essential surface,
then there is a **normal** essential surface.

Normal surface coordinates



Matching equations



$$t + q \quad = \quad t' + q'$$

$$\begin{cases} t + q = t' + q' \\ \dots \\ t, t', \dots, q, q', \dots \geq 0 \end{cases}$$

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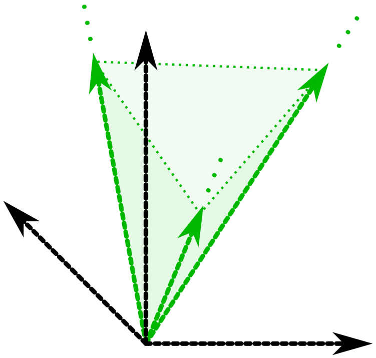
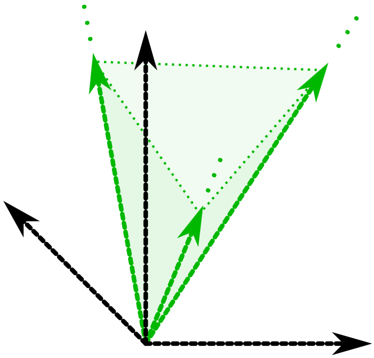


Positive rational cone

$$\begin{cases} t + q = t' + q' \\ \dots \\ t, t', \dots, q, q', \dots \geq 0 \end{cases}$$



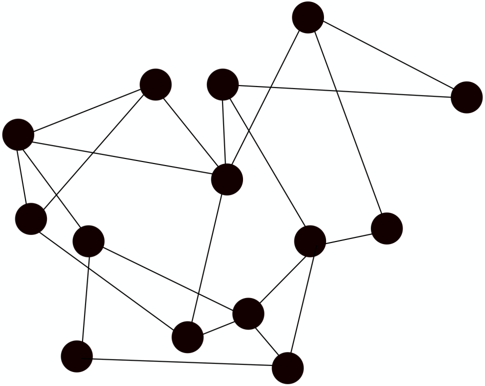
Prototheorem

If there is an essential surface,
then there is one lying on an extremal ray—
there is a **vertex-normal** essential surface.

Fact

Hyperbolicity is decidable for links in S^3.

3-Coloring



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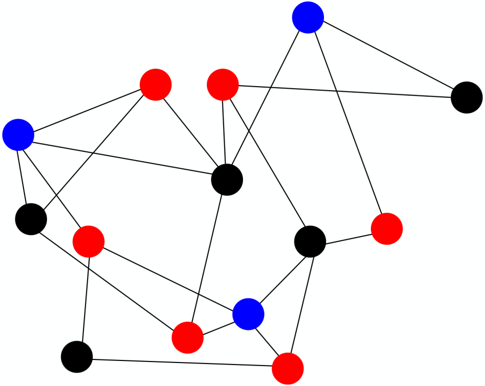
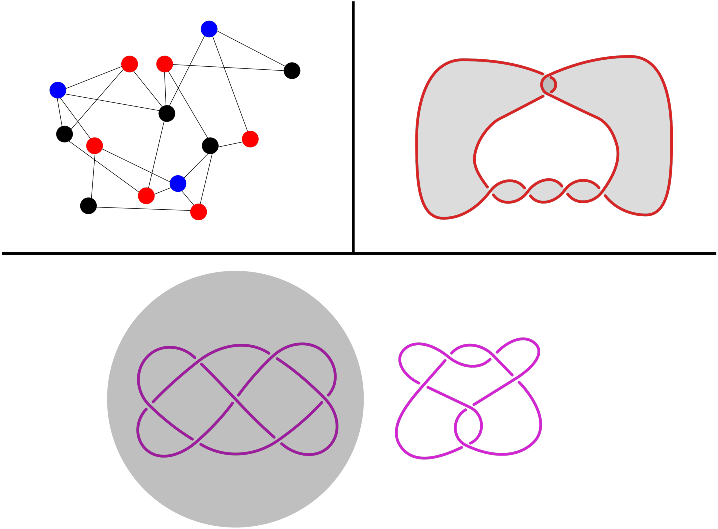


Certificates



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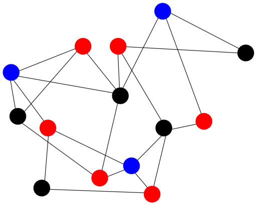

1 1 0 0 0 1 0

1 0 0 0 0 0 0 ; 1 0 0 0 0 0 0 ; 0 0 0 0 0 1 0 ; 0 0 0 0 0 0 0 ;
1 1 0 0 0 0 0 ; 0 1 0 0 0 0 0 ; 0 1 0 0 0 0 0

Workhorse Lemma

Lemma (Hass-Lagarias-Pippenger '99, 6.1.1)

Each coordinate of a vertex-normal surface in a triangulation with t *tetrahedra is at most* 2^{7t-1}.

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Unknot recognition is in NP.

That is, for every diagram K of the unknot, there is a certificate of this fact verifiable in time polynomial in $cr(K)$.

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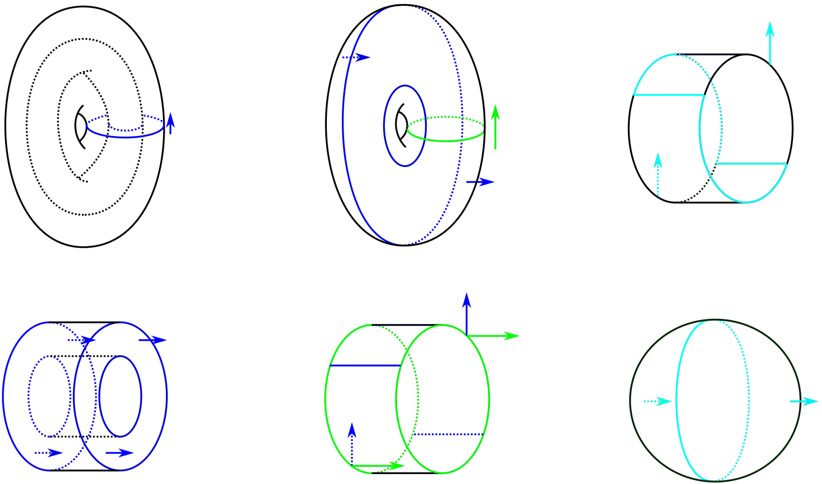
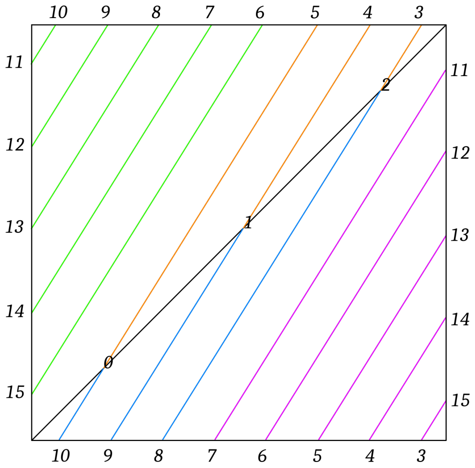
Lackenby's work



Agol-Hass-Thurston machinery



[6, 10] ↔ [15, 11]   [0, 2] ↔ [5, 3]

[0, 2] ↔ [10, 8]   [3, 7] ↔ [15, 11]

↓

one orbit

$T^2 \times I$-recognition on a napkin

Algorithm 1 Is $\mathcal{T} \approx T^2 \times I$?

procedure $T^2 \times I?(\mathcal{T})$
    if $\mathcal{T}$ is not a homology $T^2 \times I$ **then**
        return false
    Let κ be a boundary component of $\mathcal{T}$.
    Change $\mathcal{T}$ so κ has one vertex.
    for each edge e of κ **do**
        Let $\mathcal{T}_e$ be $\mathcal{T}$ "folded" along e.
        if $\mathcal{T}_e$ is not a solid torus **then**
            return false
    return true

$20 >$